There is indeed great satisfaction in acquiring skill, in coming to thoroughly understand the qualities of the material at hand and in learning to use the instruments we have—in the first place, our hands!—in an effective and controlled way.

M. C. ESCHER

Drawing Hands, M. C. Escher, 1948
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OBJECTIVES

In this chapter you will

- learn about the history of geometric constructions
- develop skills using a compass, a straightedge, patty paper, and geometry software
- see how to create complex figures using only a compass, a straightedge, and patty paper
- explore points of concurrency in triangles

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Duplicating Segments and Angles

The compass, like the straightedge, has been a useful geometry tool for thousands of years. The ancient Egyptians used the compass to mark off distances. During the Golden Age of Greece, Greek mathematicians made a game of geometric constructions. In his work *Elements*, Euclid (325–265 B.C.E.) established the basic rules for constructions using only a compass and a straightedge. In this course you will learn how to construct geometric figures using these tools as well as patty paper.

Constructions with patty paper are a variation on the ancient Greek game of geometric constructions. Almost all the figures that can be constructed with a compass and a straightedge can also be constructed using a straightedge and patty paper, waxed paper, or tracing paper. If you have access to a computer with a geometry software program, you can do constructions electronically.

In the previous chapters, you drew and sketched many figures. In this chapter, however, you'll construct geometric figures. The words sketch, draw, and construct have specific meanings in geometry.

Euclidean geometry is the study of geometry based on the assumptions of Euclid (325–265 B.C.E.). Euclid established the basic rules for constructions using only a compass and a straightedge. In his work *Elements*, Euclid proposed definitions and constructions about points, lines, angles, surfaces, and solids. He also explained why the constructions were correct with deductive reasoning.

A page from a book on Euclid, above, shows some of his constructions and a translation of his explanations from Greek into Latin.

When you sketch an equilateral triangle, you may make a freehand sketch of a triangle that looks equilateral. You don't need to use any geometry tools.

When you draw an equilateral triangle, you should draw it carefully and accurately, using your geometry tools. You may use a protractor to measure angles and a ruler to measure the sides to make sure they are equal in measure.
When you *construct* an equilateral triangle with a compass and straightedge, you don’t rely on measurements from a protractor or ruler. You must use only a compass and a straightedge. This method of construction guarantees that your triangle is equilateral.

When you sketch or draw, use the special marks that indicate right angles, parallel segments, and congruent segments and angles.

By tradition, neither a ruler nor a protractor is ever used to perform geometric constructions, because no matter how precise we try to be, measurement always involves some amount of inaccuracy. Rulers and protractors are measuring tools, not construction tools. You may use a ruler as a straightedge in constructions, provided you do not use its marks for measuring. In the next two investigations you will discover how to duplicate a line segment and an angle using only your compass and straightedge, or using only patty paper and a straightedge. By *duplicate*, we mean to copy using construction tools.

**Investigation 1**

**Duplicating a Segment**

You will need
- a compass
- a straightedge
- a ruler
- patty paper

Step 1

The complete construction for copying a segment, $AB$, is shown above. Describe each stage of the process.

Step 2

Use a ruler to measure $AB$ and $CD$. How do the two segments compare?

Step 3

Describe how to duplicate a segment using patty paper instead of a compass.
Using only a compass and a straightedge, how would you duplicate an angle? In other words, how would you construct an angle that is congruent to a given angle? You may not use your protractor, because a protractor is a measuring tool, not a construction tool.

**Investigation 2**

**Duplicating an Angle**

**You will need**
- a compass
- a straightedge

**Step 1**
The first two stages for copying $\angle DEF$ are shown below. Describe each stage of the process.

**Stage 1**

**Stage 2**

**Step 2**
What will be the final stage of the construction?

**Step 3**
Use a protractor to measure $\angle DEF$ and $\angle G$. What can you state about these angles?

**Step 4**
Describe how to duplicate an angle using patty paper instead of a compass.

You've just discovered how to duplicate segments and angles using a straightedge and compass or patty paper. These are the basic constructions. You will use combinations of these to do many other constructions. You may be surprised that you can construct figures more precisely without using a ruler or protractor!

Called vintas, these canoes with brightly patterned sails are used for fishing in Zamboanga, Philippines. What angles and segments are duplicated in this photo?
Lesson 3.1 Duplicating Segments and Angles

**Exercises**

Construction  Now that you can duplicate line segments and angles using construction tools, do the constructions in Exercises 1–10. You will duplicate polygons in Exercises 7 and 10.

1. Using only a compass and a straightedge, duplicate the three line segments shown below. Label them as they’re labeled in the figures.

   ![Diagram of line segments A, B, C, D, E, F]

2. Use the segments from Exercise 1 to construct a line segment with length $AB + CD$.

3. Use the segments from Exercise 1 to construct a line segment with length $2AB + 2EF - CD$.

4. Use a compass and a straightedge to duplicate each angle. There’s an arc in each angle to help you.

   ![Diagram of angles]

5. Draw an obtuse angle. Label it $LGE$, then duplicate it.

6. Draw two acute angles on your paper. Construct a third angle with a measure equal to the sum of the measures of the first two angles. Remember, you cannot use a protractor—use a compass and a straightedge only.

7. Draw a large acute triangle on the top half of your paper. Duplicate it on the bottom half, using your compass and straightedge. Do not erase your construction marks, so others can see your method.

8. Construct an equilateral triangle. Each side should be the length of this segment.

9. Repeat Exercises 7 and 8 using constructions with patty paper.

10. Draw quadrilateral $QUAD$. Duplicate it, using your compass and straightedge. Label the construction $COPY$ so that $QUAD \cong COPY$.

11. Technology  Use geometry software to construct an equilateral triangle. Drag each vertex to make sure it remains equilateral.
12. Copy the diagram at right. Use the Vertical Angles Conjecture and the Parallel Lines Conjecture to calculate the measure of each angle.

13. Hyacinth is standing on the curb waiting to cross 24th Street. A half block to her left is Avenue J, and Avenue K is a half block to her right. Numbered streets run parallel to one another and are all perpendicular to lettered avenues. If Avenue P is the northernmost avenue, which direction (north, south, east, or west) is she facing?

14. Write a new definition for an isosceles triangle, based on the triangle’s reflectional symmetry. Does your definition apply to equilateral triangles? Explain.

15. Sketch the three-dimensional figure formed by folding this net into a solid.

16. Draw \( \triangle DAY \) after it is rotated 90° clockwise about the origin. Label the coordinates of the vertices.

17. Use your ruler to draw a triangle with side lengths 8 cm, 10 cm, and 11 cm. Explain your method. Can you draw a second triangle with the same three side lengths that is not congruent to the first?

**IMPROVING YOUR ALGEBRA SKILLS**

**Pyramid Puzzle II**

Place four different numbers in the bubbles at the vertices of each pyramid so that the two numbers at the ends of each edge add to the number on that edge.
To be successful, the first thing to do is to fall in love with your work.

SISTER MARY LAURETTA

LESSON 3.2

Constructing Perpendicular Bisectors

Each segment has exactly one midpoint. A segment bisector is a line, ray, or segment that passes through the midpoint of a segment.

A segment has many perpendiculars and many bisectors, but in a plane each segment has only one bisector that is also perpendicular to the segment. This line is its perpendicular bisector.

The construction of the perpendicular bisector of a segment creates a line of symmetry. You use this property when you hang a picture frame. If you want to center a picture above your desk, you need to place a nail in the wall somewhere along the perpendicular bisector of the segment that forms the top edge of your desk closest to the wall.

In this investigation you will discover how to construct the perpendicular bisector of a segment.

You will need
- patty paper
- a straightedge

Step 1
Draw a segment on patty paper. Label it $PQ$.

Step 2
Fold your patty paper so that endpoints $P$ and $Q$ land exactly on top of each other, that is, they coincide. Crease your paper along the fold.

Step 3
Unfold your paper. Draw a line in the crease. What is the relationship of this line to $PQ$? Check with others in your group. Use your ruler and protractor to verify your observations.
How would you describe the relationship of the points on the perpendicular bisector to the endpoints of the bisected segment? There’s one more step in your investigation.

Place three points on your perpendicular bisector. Label them $A$, $B$, and $C$. With your compass, compare the distances $PA$ and $QA$. Compare the distances $PB$ and $QB$. Compare the distances $PC$ and $QC$. What do you notice about the two distances from each point on the perpendicular bisector to the endpoints of the segment? Compare your results with the results of others. Then copy and complete the conjecture.

**Perpendicular Bisector Conjecture**

If a point is on the perpendicular bisector of a segment, then it is ___ from the endpoints.

You’ve just completed the Perpendicular Bisector Conjecture. What about the converse of this statement?

**Investigation 2**

**Constructing the Perpendicular Bisector**

If a point is **equidistant**, or the same distance, from two endpoints of a line segment in a plane, will it be on the segment’s perpendicular bisector? If so, then locating two such points can help you construct the perpendicular bisector.

1. **Step 1**
   - Draw a line segment. Set your compass to more than half the distance between the endpoints. Using one endpoint as center, swing an arc on one side of the segment.

2. **Step 2**
   - Using the same compass setting, but using the other endpoint as center, swing a second arc intersecting the first.

3. **Step 3**
   - The point where the two arcs intersect is equidistant from the endpoints of your segment. Just as you did on one side of the segment, use your compass to find another such point. Use these points to construct a line. Is this line the perpendicular bisector of the segment? Use the paper-folding technique of Investigation 1 to check.
Notice that constructing the perpendicular bisector also locates the midpoint of a segment. Now that you know how to construct the perpendicular bisector and the midpoint, you can construct rectangles, squares, and right triangles. You can also construct two special segments in any triangle: medians and midsegments.

The segment connecting the vertex of a triangle to the midpoint of its opposite side is a **median**. There are three midpoints and three vertices in every triangle, so every triangle has three medians.

The segment that connects the midpoints of two sides of a triangle is a **midsegment**. A triangle has three sides, each with its own midpoint, so there are three midsegments in every triangle.

**Exercises**

For Exercises 1–5, construct the figures using only a compass and a straightedge.

1. Draw and label \( AB \). Construct the perpendicular bisector of \( AB \).

2. Draw and label \( QD \). Construct perpendicular bisectors to divide \( QD \) into four congruent segments.

3. Draw a line segment so close to the edge of your paper that you can swing arcs on only one side of the segment. Then construct the perpendicular bisector of the segment.

4. Using \( AB \) and \( CD \), construct a segment with length \( 2 \frac{1}{2} CD \).

5. Construct \( MN \) with length equal to the average length of \( AB \) and \( CD \) above.
6. **Construction**  Do Exercises 1–5 using patty paper.

**Construction**  For Exercises 7–10, you have your choice of construction tools. Use either a compass and a straightedge, or patty paper and a straightedge. Do not use patty paper and compass together.

7. Construct \( \triangle ALI \). Construct the perpendicular bisector of each side. What do you notice about the three bisectors?

8. Construct \( \triangle ABC \). Construct medians \( \overline{AM} \), \( \overline{BN} \), and \( \overline{CL} \). Notice anything special?

9. Construct \( \triangle DEF \). Construct midsegment \( \overline{GH} \) where \( G \) is the midpoint of \( \overline{DF} \) and \( H \) is the midpoint of \( \overline{DE} \). What do you notice about the relationship between \( \overline{EF} \) and \( \overline{GH} \)?

10. Copy rectangle \( DSMO \) onto your paper. Construct the midpoint of each side. Label the midpoint of \( DS \) point \( I \), the midpoint of \( SO \) point \( C \), the midpoint of \( OE \) point \( V \), and the midpoint of \( ED \) point \( R \). Construct quadrilateral \( RICV \). Describe \( RICV \).

11. The island shown at right has two post offices. The postal service wants to divide the island into two zones so that anyone within each zone is always closer to their own post office than to the other one. Copy the island and the locations of the post offices and locate the dividing line between the two zones. Explain how you know this dividing line solves the problem. Or pick several points in each zone and make sure they are closer to that zone’s post office than they are to the other one.

12. Copy parallelogram \( DFAT \) onto your paper. Construct the perpendicular bisector of each side. What do you notice about the quadrilateral formed by the four lines?

13. **Technology**  Use geometry software to construct a triangle. Construct a median. Are the two triangles created by the median congruent? Use an area measuring tool in your software program to find the areas of the two triangles. How do they compare? If you made the original triangle from heavy cardboard, and you wanted to balance that cardboard triangle on the edge of a ruler, what would you do?

14. **Construction**  Construct a very large triangle on a piece of cardboard or mat board and construct its median. Cut out the triangle and see if you can balance it on the edge of a ruler. Sketch how you placed the triangle on the ruler. Cut the triangle into two pieces along the median and weigh the two pieces. Are they the same weight?
In Exercises 15–20, match the term with its figure below.

15. Scalene acute triangle
16. Isosceles obtuse triangle
17. Isosceles right triangle
18. Isosceles acute triangle
19. Scalene obtuse triangle
20. Scalene right triangle

21. List the letters from the alphabet below that have a horizontal line of symmetry.
   A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

22. Use your ruler and protractor to draw a triangle with angle measures 40° and 70° and a side opposite the 70° angle with length 10 cm. Explain your method. Can you draw a second triangle using the same instructions that is not congruent to the first?
Constructing Perpendiculars to a Line

If you are in a room, look over at one of the walls. What is the distance from where you are to that wall? How would you measure that distance? There are a lot of distances from where you are to the wall, but in geometry when we speak of a distance from a point to a line we mean the perpendicular distance.

The construction of a perpendicular from a point to a line (with the point not on the line) is another of Euclid’s constructions, and it has practical applications in many fields, including agriculture and engineering. For example, think of a high-speed Internet cable as a line and a building as a point not on the line. Suppose you wanted to connect the building to the Internet cable using the shortest possible length of connecting wire. How can you find out how much wire you need, so you don’t buy too much?

Investigation 1
Finding the Right Line

You already know how to construct perpendicular bisectors of segments. You can use that knowledge to construct a perpendicular from a point to a line.

Step 1
Draw a line and a point labeled $P$ not on the line, as shown above.

Step 2
Describe the construction steps you take at Stage 2.

Step 3
How is $PA$ related to $PB$? What does this answer tell you about where point $P$ lies? Hint: See the Converse of the Perpendicular Bisector Conjecture.

Step 4
Construct the perpendicular bisector of $AB$. Label the midpoint $M$. 
LESSON 3.3 Constructing Perpendiculars to a Line

In Investigation 1, you constructed a perpendicular from a point to a line. Now let’s do the same construction using patty paper.

On a piece of patty paper, perform the steps below.

**Step 1**
Draw and label $AB$ and a point $P$ not on $AB$.

**Step 2**
Fold the line onto itself, and slide the layers of paper so that point $P$ appears to be on the crease. Is the crease perpendicular to the line? Check it with the corner of a piece of patty paper.

**Step 3**
Label the point of intersection $M$. Are $\angle AMP$ and $\angle BMP$ congruent? Supplementary? Why or why not?

You have now constructed a perpendicular through a point not on the line. This is useful for finding the distance to a line.

Label three randomly placed points on $\overline{AB}$ as $Q$, $R$, and $S$. Measure $PQ$, $PR$, $PS$, and $PM$. Which distance is shortest? Compare results with those of others in your group.

You are now ready to state your observations by completing the conjecture.

**Shortest Distance Conjecture**

The shortest distance from a point to a line is measured along the ? from the point to the line.

Let’s take another look. How could you use patty paper to do this construction?

**Investigation 2**

**Patty-Paper Perpendiculars**

In Investigation 1, you constructed a perpendicular from a point to a line. Now let’s do the same construction using patty paper.

On a piece of patty paper, perform the steps below.

**Step 1**
Draw and label $\overline{AB}$ and a point $P$ not on $\overline{AB}$.

**Step 2**
Fold the line onto itself, and slide the layers of paper so that point $P$ appears to be on the crease. Is the crease perpendicular to the line? Check it with the corner of a piece of patty paper.

**Step 3**
Label the point of intersection $M$. Are $\angle AMP$ and $\angle BMP$ congruent? Supplementary? Why or why not?

In Investigation 2, is $M$ the midpoint of $\overline{AB}$? Do you think it needs to be? Think about the techniques used in the two investigations. How do the techniques differ?
The construction of a perpendicular from a point to a line lets you find the shortest distance from a point to a line. The geometry definition of distance from a point to a line is based on this construction, and it reads, “The distance from a point to a line is the length of the perpendicular segment from the point to the line.”

You can also use this construction to find an altitude of a triangle. An altitude of a triangle is a perpendicular segment from a vertex to the opposite side or to a line containing the opposite side.

The length of the altitude is the height of the triangle. A triangle has three different altitudes, so it has three different heights.

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The length of the altitude is the height of the triangle. A triangle has three different altitudes, so it has three different heights.

**Exercises**

Use your compass and straightedge and the definition of distance to do Exercises 1–5.

1. Draw an obtuse angle $\angle BIG$. Place a point $P$ inside the angle. Now construct perpendiculars from the point to both sides of the angle. Which side is closer to point $P$?

2. Draw an acute triangle. Label it $\triangle ABC$. Construct altitude $CD$ with point $D$ on $AB$. (We didn’t forget about point $D$. It’s at the foot of the perpendicular. Your job is to locate it.)

3. Draw obtuse triangle $\triangle OBT$ with obtuse angle $O$. Construct altitude $BU$. In an obtuse triangle, an altitude can fall outside the triangle. To construct an altitude from point $B$ of your triangle, extend side $OT$. In an obtuse triangle, how many altitudes fall outside the triangle and how many fall inside the triangle?  

4. How can you construct a perpendicular to a line through a point that is on the line? 
   Draw a line. Mark a point on your line. Now experiment. Devise a method to construct a perpendicular to your line at the point.

5. Draw a line. Mark two points on the line and label them $Q$ and $R$. Now construct a square $\square SQRE$ with $QR$ as a side.
Construction For Exercises 6–9, use patty paper and a straightedge. (Attach your patty-paper work to your problems.)

6. Draw a line across your patty paper with a straightedge. Place a point $P$ not on the line, and fold the perpendicular to the line through the point $P$. How would you fold to construct a perpendicular through a point on a line? Place a point $Q$ on the line. Fold a perpendicular to the line through point $Q$. What do you notice about the two folds?

7. Draw a very large acute triangle on your patty paper. Place a point inside the triangle. Now construct perpendiculars from the point to all three sides of the triangle by folding. Mark your figure. How can you use your construction to decide which side of the triangle your point is closest to?

8. Construct an isosceles right triangle. Label its vertices $A$, $B$, and $C$, with point $C$ the right angle. Fold to construct the altitude $CD$. What do you notice about this line?

9. Draw obtuse triangle $OBT$ with angle $O$ obtuse. Fold to construct the altitude $BU$. (Don’t forget, you must extend the side $OT$.)

Construction For Exercises 10–12, you may use either patty paper or a compass and a straightedge.

10. Construct a square $ABLE$ given $AL$ as a diagonal.

11. Construct a rectangle whose width is half its length.

12. Construct the complement of $\angle A$.

Review

13. Copy and complete the table. Make a conjecture for the value of the $n$th term and for the value of the 35th term.

<table>
<thead>
<tr>
<th>Rectangular pattern with triangles</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$\ldots$</th>
<th>$n$</th>
<th>$\ldots$</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of shaded triangles</td>
<td>2</td>
<td>9</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>$\ldots$</td>
<td>$n$</td>
<td>$\ldots$</td>
<td>35</td>
</tr>
</tbody>
</table>

14. Sketch the solid of revolution formed when the two-dimensional figure at right is revolved about the line.
For Exercises 15–20, label the vertices with the appropriate letters. When you sketch or draw, use the special marks that indicate right angles, parallel segments, and congruent segments and angles.

15. Sketch obtuse triangle $FIT$ with $m \angle I > 90^\circ$ and median $IY$.
16. Sketch $AB \perp CD$ and $EF \perp CD$.

17. Use your protractor to draw a regular pentagon. Draw all the diagonals. Use your compass to construct a regular hexagon. Draw three diagonals connecting alternating vertices. Do the same for the other three vertices.

18. Draw a triangle with a 6 cm side and an 8 cm side and the angle between them measuring 40°. Draw a second triangle with a 6 cm side and an 8 cm side and exactly one 40° angle that is not between the two given sides. Are the two triangles congruent?

19. Sketch and label a polygon that has exactly three sides of equal length and exactly two angles of equal measure.

20. Sketch two triangles. Each should have one side measuring 5 cm and one side measuring 9 cm, but they should not be congruent.

**CONSTRUCTING A TILE DESIGN**

This Islamic design is based on two intersecting squares that form an 8-pointed star. Many designs of this kind can be constructed using only patty paper or a compass and a straightedge. Try it. Use construction tools to re-create this design or to create a design of your own based on an 8-pointed star.

Your project should include

- Your design based on an 8-pointed star, in color.
- A diagram showing your construction technique, with a written explanation of how you created it.

Here is a diagram to get you started.
**Constructing Angle Bisectors**

Challenges make you discover things about yourself that you never really knew.

CICELY TYSON

**Lesson 3.4**

**Angle Bisecting by Folding**

Each person should draw his or her own acute angle for this investigation.

**Step 1**

On patty paper, draw a large-scale angle. Label it \(\triangle PQR\).

**Step 2**

Fold your patty paper so that \(\overline{QP}\) and \(\overline{QR}\) coincide. Crease the fold.

**Step 3**

Unfold your patty paper. Draw a ray with endpoint \(Q\) along the crease. Does the ray bisect \(\angle PQR\)? How can you tell?

**Step 4**

Repeat Steps 1–3 with an obtuse angle. Do you use different methods for finding the bisectors of different kinds of angles?

**Step 5**

Place a point on your angle bisector. Label it \(A\). Compare the distances from \(A\) to each of the two sides. Remember that “distance” means shortest distance! Try it with other points on the angle bisector. Compare your results with those of others. Copy and complete the conjecture.

**Angle Bisector Conjecture**

If a point is on the bisector of an angle, then it is \(\frac{1}{2}\) from the sides of the angle.
You’ve found the bisector of an angle by folding patty paper. Now let’s see how you can construct the angle bisector with a compass and a straightedge.

**Investigation 2**

**Angle Bisecting with Compass**

In this investigation, you will find a method for bisecting an angle using a compass and straightedge. Each person in your group should investigate a different angle.

**Step 1**
Draw an angle.

**Step 2**
Find a method for constructing the bisector of the angle. Experiment!

Hint: Start by drawing an arc centered at the vertex.

**Step 3**
Once you think you have constructed the angle bisector, fold your paper to see if the ray you constructed is actually the bisector. Share your method with other students in your group. Agree on a best method.

**Step 4**
Write a summary of what you did in this investigation.

In earlier lessons, you learned to construct a 90° angle. Now you know how to bisect an angle. What angles can you construct by combining these two skills?

**EXERCISES**

*Construction*

For Exercises 1–5, match each geometric construction with its diagram.

1. Construction of an angle bisector
2. Construction of a median
3. Construction of a midsegment
4. Construction of a perpendicular bisector
5. Construction of an altitude

![Diagrams of geometric constructions](image-url)
Construction  For Exercises 6–12, construct a figure with the given specifications.

6. Given:

   \[ \overline{AB} \]

   **Construct:** An isosceles right triangle with \( z \) as the length of each of the two congruent sides

7. Given:

   \[ \overline{AR} \quad \overline{RP} \]

   **Construct:** \( \triangle RAP \) with median \( \overline{PM} \) and angle bisector \( \overline{RB} \)

8. Given:

   \[ \overline{MO} \quad \overline{SE} \]

   **Construct:** \( \triangle MSE \) with \( \overline{OU} \) where \( O \) is the midpoint of \( \overline{MS} \) and \( U \) is the midpoint of \( \overline{SE} \)

9. Construct an angle with each given measure and label it. Remember, you may use only your compass and straightedge. No protractor!
   
a. \( 90^\circ \)  
b. \( 45^\circ \)  
c. \( 135^\circ \)

10. Draw a large acute triangle. Bisect the angle at one vertex with a compass and a straightedge. Construct an altitude from the second vertex and a median from the third vertex.

11. Repeat Exercise 10 with patty paper. Which set of construction tools do you prefer? Why?

12. Use your straightedge to construct a linear pair of angles. Use your compass to bisect each angle of the linear pair. What do you notice about the two angle bisectors? Can you make a conjecture? Can you explain why it is true?

13. In this lesson you discovered the Angle Bisector Conjecture. Write the converse of the Angle Bisector Conjecture. Do you think it’s true? Why or why not?

Notice how this mosaic floor at Church of Pomposa in Italy (ca. 850 C.E.) uses many duplicated shapes. What constructions do you see in the square pattern? Are all the triangles in the isosceles triangle pattern identical? How can you tell?

15. If $\overline{AE}$ bisects $\angle CAR$ and $m \angle CAR = 84^\circ$, find $m \angle R$.  

16. Which angle is largest, $\angle A$, $\angle B$, or $\angle C$?  

Review

Draw or construct each figure in Exercises 17–21. Label the vertices with the appropriate letters. If you’re unclear on the difference between “draw” and “construct,” refer back to pages 144 and 145.

17. Draw a regular octagon. What traffic sign comes to mind?  

18. Construct regular octagon $\text{ALTOSIGN}$.  

19. Draw $\triangle ABC$ so that $AC = 3.5$ cm, $AB = 5.6$ cm, and $m \angle BAC = 130^\circ$.  

20. Draw isosceles right $\triangle ABC$ so that $BC = 6.5$ cm and $m \angle B = 90^\circ$.  

21. Draw a triangle with a 40° angle, a 60° angle, and a side between the given angles measuring 8 cm. Draw a second triangle with a 40° angle and a 60° angle but with a side measuring 8 cm opposite the 60° angle. Are the triangles congruent?  

22. Technology Use geometry software to construct $\overline{AB}$ and $\overline{CD}$, with point $C$ on $\overline{AB}$ and point $D$ not on $\overline{AB}$. Construct the perpendicular bisector of $\overline{CD}$.

   a. Trace this perpendicular bisector as you drag point $C$ along $\overline{AB}$. Describe the shape formed by this locus of lines.

   b. Erase the tracings from part a. Now trace the midpoint of $\overline{CD}$ as you drag $C$. Describe the locus of points.

IMPROVING YOUR VISUAL THINKING SKILLS

Coin Swap III

Arrange four dimes and four pennies in a row of nine squares, as shown. Switch the position of the four dimes and four pennies in exactly 24 moves. A coin can slide into an empty square next to it or can jump over one coin into an empty space. Record your solution by listing, in order, which type of coin is moved. For example, your list might begin PDPDPDPPDD . . . .
Constructing Parallel Lines

Parallel lines are lines that lie in the same plane and do not intersect.

The lines in the first pair shown above intersect. They are clearly not parallel. The lines in the second pair do not meet as drawn. However, if they were extended, they would intersect. Therefore, they are not parallel. The lines in the third pair appear to be parallel, but if you extend them far enough in both directions, can you be sure they won’t meet? There are many ways to be sure that the lines are parallel.

### Constructing Parallel Lines by Folding

How would you check whether two lines are parallel? One way is to draw a transversal and compare corresponding angles. You can also use this idea to construct a pair of parallel lines.

**You will need**
- patty paper
- a straightedge

1. **Step 1**
   Draw a line and a point on patty paper as shown.

2. **Step 2**
   Fold the paper to construct a perpendicular so that the crease runs through the point as shown. Describe the four newly formed angles.

3. **Step 3**
   Through the point, make another fold that is perpendicular to the first crease.

4. **Step 4**
   Compare the pairs of corresponding angles created by the folds. Are they all congruent? Why? What conclusion can you make about the lines?
There are many ways to construct parallel lines. You can construct parallel lines much more quickly with patty paper than with compass and straightedge. You can also use properties you discovered in the Parallel Lines Conjecture to construct parallel lines by duplicating corresponding angles, alternate interior angles, or alternate exterior angles. Or you can construct two perpendiculars to the same line. In the exercises you will practice all of these methods.

**EXERCISES**

**Construction** In Exercises 1–9, use the specified construction tools to do each construction. If no tools are specified, you may choose either patty paper or compass and straightedge.

1. Use compass and straightedge. Draw a line and a point not on the line. Construct a second line through the point that is parallel to the first line, by duplicating alternate interior angles.

2. Use compass and straightedge. Draw a line and a point not on the line. Construct a second line through the point that is parallel to the first line, by duplicating corresponding angles.

3. Construct a square with perimeter $z$.

4. Construct a rhombus with $x$ as the length of each side and $\angle A$ as one of the acute angles.

5. Construct trapezoid $TRAP$ with $\overline{TR}$ and $\overline{AP}$ as the two parallel sides and with $\overline{AP}$ as the distance between them. (There are many solutions!)

6. Using patty paper and straightedge, or a compass and straightedge, construct parallelogram $GRAM$ with $\overline{GR}$ and $\overline{RA}$ as two consecutive sides and $\overline{ML}$ as the distance between $\overline{GR}$ and $\overline{AM}$. (How many solutions can you find?)
You may choose to do the mini-investigations in Exercises 7, 9, and 11 using geometry software.

7. **Mini-Investigation** Draw a large scalene acute triangle and label it $\triangle SUM$. Through vertex $M$ construct a line parallel to side $SU$ as shown in the diagram. Use your protractor or a piece of patty paper to compare $\angle 1$ and $\angle 2$ with the other two angles of the triangle ($\angle S$ and $\angle U$). Notice anything special? Write down what you observe.

8. **Developing Proof** Use deductive reasoning to explain why your observation in Exercise 7 is true for any triangle.

9. **Mini-Investigation** Draw a large scalene acute triangle and label it $\triangle PAR$. Place point $E$ anywhere on side $PR$, and construct a line $\overline{EL}$ parallel to side $PA$ as shown in the diagram. Use your ruler to measure the lengths of the four segments $\overline{AL}$, $\overline{LR}$, $\overline{RE}$, and $\overline{EP}$ and compare ratios $\frac{\overline{EL}}{\overline{LA}}$ and $\frac{\overline{EP}}{\overline{LA}}$. Notice anything special? Write down what you observe.

10. **Developing Proof** Measure the four labeled angles in Exercise 9. Notice anything special? Use deductive reasoning to explain why your observation is true for any triangle.

11. **Mini-Investigation** Draw a pair of parallel lines by tracing along both edges of your ruler. Draw a transversal. Use your compass to bisect each angle of a pair of alternate interior angles. What shape is formed?

12. **Developing Proof** Use deductive reasoning to explain why the resulting shape is formed in Exercise 11.

---

**Review**

13. There are three fire stations in the small county of Dry Lake. County planners need to divide the county into three zones so that fire alarms alert the closest station. Trace the county and the three fire stations onto patty paper, and locate the boundaries of the three zones. Explain how these boundaries solve the problem.

Sketch or draw each figure in Exercises 14–16. Label the vertices with the appropriate letters. Use the special marks that indicate right angles, parallel segments, and congruent segments and angles.

14. Sketch trapezoid $ZOID$ with $\overline{ZO} \parallel \overline{ID}$, point $T$ the midpoint of $\overline{OI}$, and $R$ the midpoint of $\overline{ZD}$. Sketch segment $TR$.

15. Draw rhombus $ROMB$ with $m \angle R = 60^\circ$ and diagonal $\overline{OB}$.

16. Draw rectangle $RECK$ with diagonals $\overline{RC}$ and $\overline{EK}$ both 8 cm long and intersecting at point $W$. 

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17. **Developing Proof** Copy the diagram below. Use your conjectures to calculate the measure of each lettered angle. Explain how you determined measures $m, p,$ and $r.$

\[ \text{Diagram with angles labeled } h, i, j, k, d, g, c, m, n, p, q, \text{ and } r. \]

**IMPROVING YOUR VISUAL THINKING SKILLS**

**Visual Analogies**

Which of the designs at right complete the statements at left? Explain.

1. [Diagram with two sets of shapes and a question mark]
   - A. [Design A]
   - B. [Design B]
   - C. [Design C]
   - D. [Design D]

2. [Diagram with two sets of shapes and a question mark]
   - A. [Design A]
   - B. [Design B]
   - C. [Design C]
   - D. [Design D]

3. [Diagram with two sets of shapes and a question mark]
   - A. [Design A]
   - B. [Design B]
   - C. [Design C]
   - D. [Design D]
Slopes of Parallel and Perpendicular Lines

If two lines are parallel, how do their slopes compare? If two lines are perpendicular, how do their slopes compare? In this lesson you will review properties of the slopes of parallel and perpendicular lines.

If the slopes of two or more distinct lines are equal, are the lines parallel? To find out, try drawing on graph paper two lines that have the same slope triangle.

Yes, the lines are parallel. In fact, in coordinate geometry, this is the definition of parallel lines. The converse of this is true as well: If two lines are parallel, their slopes must be equal.

**Parallel Slope Property**

In a coordinate plane, two distinct lines are parallel if and only if their slopes are equal, or they are both vertical lines.

If two lines are perpendicular, their slope triangles have a different relationship. Study the slopes of the two perpendicular lines at right.

**Perpendicular Slope Property**

In a coordinate plane, two nonvertical lines are perpendicular if and only if their slopes are opposite reciprocals of each other.

Can you explain why the slopes of perpendicular lines would have opposite signs? Can you explain why they would be reciprocals? Why do the lines need to be nonvertical?
**EXAMPLE A**

Consider $A(-15,-6)$, $B(6,8)$, $C(4,-2)$ and $D(-4,10)$. Are $\overline{AB}$ and $\overline{CD}$ parallel, perpendicular, or neither?

**Solution**

Calculate the slope of each line.

\[
\text{slope of } \overline{AB} = \frac{8 - (-6)}{6 - (-15)} = \frac{2}{3}
\]

\[
\text{slope of } \overline{CD} = \frac{10 - (-2)}{-4 - (-4)} = -\frac{3}{2}
\]

The slopes, $\frac{2}{3}$ and $-\frac{3}{2}$, are opposite reciprocals of each other, so $\overline{AB} \perp \overline{CD}$.

**EXAMPLE B**

Given points $E(-3,0)$, $F(5,-4)$, and $Q(4,2)$, find the coordinates of a point $P$ such that $\overline{PQ}$ is parallel to $\overline{EF}$.

**Solution**

We know that if $\overline{PQ} \parallel \overline{EF}$, then the slope of $\overline{PQ}$ equals the slope of $\overline{EF}$. First find the slope of $\overline{EF}$.

\[
\text{slope of } \overline{EF} = \frac{-4 - 0}{5 - (-3)} = \frac{-4}{8} = -\frac{1}{2}
\]

There are many possible ordered pairs $(x, y)$ for $P$. Use $(x, y)$ as the coordinates of $P$, and the given coordinates of $Q$, in the slope formula to get

\[
\frac{2 - y}{4 - x} = -\frac{1}{2}
\]

Now you can treat the denominators and numerators as separate equations.

\[
\begin{align*}
4 - x &= 2 \\
-x &= -2 \\
x &= 2
\end{align*}
\]

\[
\begin{align*}
2 - y &= -1 \\
y &= -3
\end{align*}
\]

Thus one possibility is $P(2, 3)$. How could you find another ordered pair for $P$? Here’s a hint: How many different ways can you express $-\frac{1}{2}$?

Coordinate geometry is sometimes called “analytic geometry.” This term implies that you can use algebra to further analyze what you see. For example, consider $\overline{AB}$ and $\overline{CD}$. They look parallel, but looks can be deceiving. Only by calculating the slopes will you see that the lines are not truly parallel.
EXERCISES

For Exercises 1–4, determine whether each pair of lines through the points given below is parallel, perpendicular, or neither.

\[ A(1, 2) \quad B(3, 4) \quad C(5, 2) \quad D(8, 3) \quad E(3, 8) \quad F(-6, 5) \]

1. \( AB \) and \( BC \)
2. \( AB \) and \( CD \)
3. \( AB \) and \( DE \)
4. \( CD \) and \( EF \)

5. Given \( A(0, -3), B(5, 3), \) and \( Q( -3, \ -1) \), find two possible locations for a point \( P \) such that \( PQ \) is parallel to \( AB \).

6. Given \( C( -2, \ -1), D(5, \ -4), \) and \( Q(4, 2) \), find two possible locations for a point \( P \) such that \( PQ \) is perpendicular to \( CD \).

For Exercises 7–9, find the slope of each side, and then determine whether each figure is a trapezoid, a parallelogram, a rectangle, or just an ordinary quadrilateral. Explain how you know.

7.

8.

9.

10. Quadrilateral \( HAND \) has vertices \( H(-5, \ -1), A(7, \ 1), N(6, \ 7), \) and \( D(-6,5) \).
   b. Find the midpoint of each diagonal. What can you conjecture?

11. Quadrilateral \( OVER \) has vertices \( O(-4, \ 2), V(1, \ 1), E(0, \ 6), \) and \( R(-5, \ 7) \).
   a. Are the diagonals perpendicular? Explain how you know.
   b. Find the midpoint of each diagonal. What can you conjecture?
   c. What type of quadrilateral does \( OVER \) appear to be? Explain how you know.

12. Consider the points \( A( -5, \ -2), B(1, \ 1), C( -1, \ 0), \) and \( D(3, \ 2) \).
   a. Find the slopes of \( AB \) and \( CD \).
   b. Despite their slopes, \( AB \) and \( CD \) are not parallel. Why not?
   c. What word in the Parallel Slope Property addresses the problem in 12b?

13. Given \( A(-3, \ 2), B(1, \ 5), \) and \( C(7, \ -3) \), find point \( D \) such that quadrilateral \( ABCD \) is a rectangle.
Construction Problems

Once you know the basic constructions, you can create more complex geometric figures.

You know how to duplicate segments and angles with a compass and straightedge. Given a triangle, you can use these two constructions to duplicate the triangle by copying each segment and angle. Can you construct a triangle if you are given the parts separately? Would you need all six parts—three segments and three angles—to construct a triangle?

Let’s first consider a case in which only three segments are given.

**EXAMPLE A**

Construct ΔABC using the three segments \(AB\), \(BC\), and \(CA\) shown below. How many different-size triangles can be drawn?

**Solution**

You can begin by duplicating one segment, for example \(AC\). Then adjust your compass to match the length of another segment. Using this length as a radius, draw an arc centered at one endpoint of the first segment. Now use the third segment length as the radius for another arc, this one centered at the other endpoint. Where the arcs intersect is the location of the third vertex of the triangle.

In the construction above, the segment lengths determine where the arcs intersect. Once the triangle “closes” at the intersection of the arcs, the angles are determined too. So the lengths of the segments affect the size of the angles.

There are other ways to construct ΔABC. For example, you could draw the arcs below \(AC\) and point \(B\) would be below \(AC\). Or you could start by duplicating \(BC\) instead of \(AC\). But if you try these constructions, you will find that they all produce congruent triangles. There is only one size of triangle that can be drawn with the segments given, so the segments determine the triangle. Does having three angles also determine a triangle?

**EXAMPLE B**

Construct ΔABC with patty paper by duplicating the three angles \(\angle A\), \(\angle B\), and \(\angle C\) shown at right. How many different size triangles can be drawn?

**Solution**

In this patty-paper construction the angles do not determine the segment length. You can locate the endpoint of a segment anywhere along an angle’s side without affecting the angle measures. As shown in the illustration at the top of the next page, the patty paper with \(\angle A\) can slide horizontally over the patty paper with \(\angle B\) to
create triangles of different sizes. (The third angle in both cases is equal to \( \angle C \).) By sliding \( \angle A \), infinitely many different triangles can be drawn with the angles given. Therefore, three angles do not determine a triangle.

Because a triangle has a total of six parts, there are several combinations of segments and angles that may or may not determine a triangle. Having one or two parts given is not enough to determine a triangle. Is having three parts enough? That answer depends on the combination. In the exercises you will construct triangles and quadrilaterals with various combinations of parts given.

**EXERCISES**

*Construction* In Exercises 1–10, first sketch and label the figure you are going to construct. Second, construct the figure, using either a compass and straightedge, or patty paper and straightedge. Third, describe the steps in your construction in a few sentences.

1. Given:

```
  M A S
  M
```

Construct: \( \triangle MAS \)

2. Given:

```
  O D T
```

Construct: \( \triangle DOT \)

3. Given:

```
  I Y
```

Construct: \( \triangle IGY \)
4. Given the triangle shown at right, construct another triangle with angles congruent to the given angles but with sides not congruent to the given sides. Is there more than one noncongruent triangle with the same three angles?

5. The two segments and the angle below do not determine a triangle.

**Given:**

[Diagram of two segments and an angle]

**Construct:** Two different (noncongruent) triangles named $ABC$ that have the three given parts.

6. **Given:**

[Diagram with lengths $x$ and $y$]

**Construct:** Isosceles triangle $CAT$ with perimeter $y$ and length of the base equal to $x$.

7. Construct a kite.

8. Construct a quadrilateral with two pairs of opposite sides of equal length.

9. Construct a quadrilateral with exactly three sides of equal length.

10. Construct a quadrilateral with all four sides of equal length.

11. **Technology** Using geometry software, draw a large scalene obtuse triangle $ABC$ with $\angle B$ the obtuse angle. Construct the angle bisector $BR$, the median $BM$, and the altitude $BS$. What is the order of the points on $AC$? Drag $B$. Is the order of points always the same? Write a conjecture.

**Art CONNECTION**

The designer of stained glass arranges pieces of painted glass to form the elaborate mosaics that you might see in Gothic cathedrals or on Tiffany lampshades. He first organizes the glass pieces by shape and color according to the design. He mounts these pieces into a metal framework that will hold the design. With precision, the designer cuts every glass piece so that it fits against the next one with a strip of cast lead. The result is a pleasing combination of colors and shapes that form a luminous design when viewed against light.
Review

12. Draw the new position of $\triangle TEA$ if it is reflected over the dotted line. Label the coordinates of the vertices.

13. Draw each figure and decide how many reflectional and rotational symmetries it has. Copy and complete the table below.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Reflectional symmetries</th>
<th>Rotational symmetries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trapezoid</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kite</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Parallelogram</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rhombus</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rectangle</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

14. Sketch the three-dimensional figure formed by folding the net at right into a solid.

15. If a polygon has 500 diagonals from each vertex, how many sides does it have?

16. Use your geometry tools to draw parallelogram $CARE$ so that $CA = 5.5$ cm, $CE = 3.2$ cm, and $m \angle A = 110^\circ$.

**IMPROVING YOUR REASONING SKILLS**

**Spelling Card Trick**

This card trick uses one complete suit (hearts, clubs, spades, or diamonds) from a deck of playing cards. How must you arrange the cards so that you can successfully complete the trick? Here is what your audience should see and hear as you perform.

1. As you take the top card off the pile and place it back underneath of the pile, say “A.”
2. Then take the second card, place it at the bottom of the pile, and say “C.”
3. Take the third card, place it at the bottom, and say “E.”
4. You’ve just spelled $ace$. Now take the fourth card and turn it faceup on the table, not back in the pile. The card should be an ace.
5. Continue in this fashion, saying “T,” “W,” and “O” for the next three cards. Then turn the next card faceup. It should be a $2$.
6. Continue spelling $three, four, \ldots, jack, queen, king$. Each time you spell a card, the next card turned faceup should be that card.
You know from experience that when you look down a long straight road, the parallel edges and the center line seem to meet at a point on the horizon. To show this effect in a drawing, artists use **perspective**, the technique of portraying solid objects and spatial relationships on a flat surface. Renaissance artists and architects in the 15th century developed perspective, turning to geometry to make art appear true-to-life.

In a perspective drawing, receding parallel lines (lines that run directly away from the viewer) converge at a **vanishing point** on the **horizon line**. Locate the horizon line, the vanishing point, and converging lines in the perspective study below by Jan Vredeman de Vries.
**Activity**

**Boxes in Space**

In this activity you'll learn to draw a box in perspective.

Perspective drawing is based on the relationships between many parallel and perpendicular lines. The lines that recede to the horizon make you visually think of parallel lines even though they actually intersect at a vanishing point.

First, you’ll draw a rectangular solid, or box, in one-point perspective. Look at the diagrams below for each step.

- **Step 1**
  - Draw a horizon line \( h \) and a vanishing point \( V \). Draw the front face of the box with its horizontal edges parallel to \( h \).

- **Step 2**
  - Connect the corners of the box face to \( V \) with dashed lines.

- **Step 3**
  - Draw the upper rear box edge parallel to \( h \). Its endpoints determine the vertical edges of the back face.

- **Step 4**
  - Draw the hidden back vertical and horizontal edges with dashed lines. Erase unnecessary lines and dashed segments.

- **Step 5**
  - Repeat Steps 1–4 several more times, each time placing the first box face in a different position with respect to \( h \) and \( V \)—above, below, or overlapping \( h \); to the left or right of \( V \) or centered on \( V \).

- **Step 6**
  - Share your drawings in your group. Tell which faces of the box recede and which are parallel to the imaginary window or picture plane that you see through. What is the shape of a receding box face? Think of each drawing as a scene. Where do you seem to be standing to view each box? That is, how is the viewing position affected by placing \( V \) to the left or right of the box? Above or below the box?
If the front surface of a box is not parallel to the picture plane, then you need two vanishing points to show the two front faces receding from view. This is called **two-point perspective**. Let’s look at a rectangular solid with one edge viewed straight on.

**Step 7** Draw a horizon line $h$ and select two vanishing points on it, $V_1$ and $V_2$. Draw a vertical segment for the nearest box edge.

**Step 8** Connect each endpoint of the box edge to $V_1$ and $V_2$ with dashed lines.

**Step 9** Draw two vertical segments within the dashed lines as shown. Connect their endpoints to the endpoints of the front edge along the dashed lines. Now you have determined the position of the hidden back edges that recede from view.
Step 10
Draw the remaining edges along vanishing lines, using dashed lines for hidden edges. Erase unnecessary dashed segments.

Step 11
Repeat Steps 7–10 several times, each time placing the nearest box edge in a different position with respect to $h$, $V_1$, and $V_2$, and varying the distance between $V_1$ and $V_2$. You can also experiment with different-shaped boxes.

Step 12
Share your drawings in your group. Are any faces of the box parallel to the picture plane? Does each box face have a pair of parallel sides?

Explain how the viewing position is affected by the distance between $V_1$ and $V_2$ relative to the size of the box. Must the box be between $V_1$ and $V_2$?

Using perspective helps in designing the lettering painted on streets. From above, letters appear tall, but from a low angle, they appear normal. Tilt the page up to your face. How do the letters look?
CHAPTER 3 Using Tools of Geometry

Nothing in life is to be feared, it is only to be understood.
MARIE CURIE

LESSON 3.7

You now can perform a number of constructions in triangles, including angle bisectors, perpendicular bisectors of the sides, medians, and altitudes. In this lesson and the next lesson you will discover special properties of these lines and segments. When three or more lines have a point in common, they are concurrent. Segments, rays, and even planes are concurrent if they intersect in a single point.

The point of intersection is the point of concurrency.

Investigation 1

You will need
- patty paper
- geometry software (optional)

Concurrence

In this investigation you will discover that some special lines in a triangle have points of concurrency.

As a group, you should investigate each set of lines on an acute triangle, an obtuse triangle, and a right triangle to be sure that your conjectures apply to all triangles.

Step 1 | Draw a large triangle on patty paper. Make sure you have at least one acute triangle, one obtuse triangle, and one right triangle in your group.

Step 2 | Construct the three angle bisectors for each triangle. Are they concurrent?

Compare your results with the results of others. State your observations as a conjecture.

Angle Bisector Concurrency Conjecture

The three angle bisectors of a triangle are concurrent.

Step 3 | Draw a large triangle on a new piece of patty paper. Make sure you have at least one acute triangle, one obtuse triangle, and one right triangle in your group.

Step 4 | Construct the perpendicular bisector for each side of the triangle and complete the conjecture.
**Construction Tools**

**Geometry Software** (optional)

**Construction Tools:**

**Geometry Software** (optional)

**Investigation 2**

**Circumcenter**

In this investigation you will discover special properties of the circumcenter.

**Step 1**

Using your patty paper from Steps 3 and 4 of the previous investigation, measure and compare the distances from the circumcenter to each of the three vertices. Are they the same? Compare the distances from the circumcenter to each of the three sides. Are they the same?

**Step 2**

Tape or glue your patty paper firmly on a piece of regular paper. Use a compass to construct a circle with the circumcenter as the center and that passes through any one of the triangle’s vertices. What do you notice?

**Step 3**

Use your observations to state your next conjecture.

**Circumcenter Conjecture**

The circumcenter of a triangle __. **
You just discovered a very useful property of the circumcenter and a very useful property of the incenter. You will see some applications of these properties in the exercises. With earlier conjectures and logical reasoning, you can explain why your conjectures are true.

### Deductive Argument for the Circumcenter Conjecture

Because the circumcenter is constructed from perpendicular bisectors, the diagram of \( \triangle LYA \) at left shows two (of the three) perpendicular bisectors, \( \ell_1 \) and \( \ell_2 \). We want to show that the circumcenter, point \( P \), is equidistant from all three vertices. In other words, we want to show that

\[
PL = PA = PY
\]

A useful reasoning strategy is to break the problem into parts. In this case, we might first think about explaining why \( PL \approx PA \). To do that, let’s simplify the diagram by looking at just the bottom triangle formed by points \( P, L, \) and \( A \).

If a point is on the perpendicular bisector of a segment, it is equidistant from the endpoints.

Point \( P \) lies on the perpendicular bisector of \( LA \)

\[
PA = PL
\]

As part of the strategy of concentrating on just part of the problem, think about explaining why \( PA \approx PT \). Focus on the triangle on the left side of \( \triangle LYA \) formed by points \( P, L, \) and \( Y \).
Point $P$ also lies on the perpendicular bisector of $LY$.

$$PL = PY$$

Therefore $P$ is equidistant from all three vertices.

$$PA = PL = PY$$

As you discovered in Investigation 2, the circumcenter is the center of a circle that passes through the three vertices of a triangle.

As you found in Investigation 3, the incenter is the center of a circle that touches each side of the triangle. Here are a few vocabulary terms that help describe these geometric situations.

A circle is **circumscribed** about a polygon if and only if it passes through each vertex of the polygon. (The polygon is inscribed in the circle.)

A circle is **inscribed** in a polygon if and only if it touches each side of the polygon at exactly one point. (The polygon is circumscribed about the circle.)

---

**Developing Proof** In your groups discuss the following two questions and then write down your answers.

1. Why does the circumcenter construction guarantee that it is the center of the circle that circumscribes the triangle?

2. Why does the incenter construction guarantee that it is the center of the circle that is inscribed in the triangle?

---

**EXERCISES**

For Exercises 1–4, make a sketch and explain how to find the answer.

1. The first-aid center of Mt. Thermopolis State Park needs to be at a point that is equidistant from three bike paths that intersect to form a triangle. Locate this point so that in an emergency, medical personnel will be able to get to any one of the paths by the shortest route possible. Which point of concurrency is it?
2. An artist wishes to circumscribe a circle about a triangle in his latest abstract design. Which point of concurrency does he need to locate?

3. Rosita wants to install a circular sink in her new triangular countertop. She wants to choose the largest sink that will fit. Which point of concurrency must she locate? Explain.

4. Julian Chive wishes to center a butcher-block table at a location equidistant from the refrigerator, stove, and sink. Which point of concurrency does Julian need to locate?

5. One event at this year’s Battle of the Classes will be a pie-eating contest between the sophomores, juniors, and seniors. Five members of each class will be positioned on the football field at the points indicated at right. At the whistle, one student from each class will run to the pie table, eat exactly one pie, and run back to his or her group. The next student will then repeat the process. The first class to eat five pies and return to home base will be the winner of the pie-eating contest. Where should the pie table be located so that it will be a fair contest? Describe how the contest planners should find that point.

6. **Construction** Draw a large triangle. Construct a circle inscribed in the triangle.

7. **Construction** Draw a triangle. Construct a circle circumscribed about the triangle.

8. Is the inscribed circle the greatest circle to fit within a given triangle? Explain. If you think not, give a counterexample.

9. Does the circumscribed circle create the smallest circular region that contains a given triangle? Explain. If you think not, give a counterexample.

10. Use geometry software to construct the circumcenter of a triangle. Drag a vertex to observe how the location of the circumcenter changes as the triangle changes from acute to obtuse. What do you notice? Where is the circumcenter located for a right triangle?

11. Use geometry software to construct the orthocenter of a triangle. Drag a vertex to observe how the location of the orthocenter changes as the triangle changes from acute to obtuse. What do you notice? Where is the orthocenter located for a right triangle?

**Review**

**Construction** Use the segments and angle at right to construct each figure in Exercises 12–15.

12. **Mini-Investigation** Construct $\triangle MAT$. Construct $H$ the midpoint of $MT$ and $S$ the midpoint of $AT$. Construct the midsegment $HS$. Compare the lengths of $HS$ and $MA$. Notice anything special?

13. **Mini-Investigation** An isosceles trapezoid is a trapezoid with the nonparallel sides congruent. Construct isosceles trapezoid $MOAT$ with $MT \parallel OA$ and $AT = MO$. Use patty paper to compare $\angle T$ and $\angle M$. Notice anything special?

14. **Mini-Investigation** Construct a circle with diameter $MT$. Construct chord $TA$. Construct chord $MA$ to form $\triangle TMA$. What is the measure of $\angle A$? Notice anything special?

15. **Mini-Investigation** Construct a rhombus with $TA$ as the length of a side and $\angle T$ as one of the acute angles. Construct the two diagonals. Notice anything special?

16. Sketch the locus of points on the coordinate plane in which the sum of the $x$-coordinate and the $y$-coordinate is 9.

17. **Construction** Bisect the missing angle of this triangle. How can you do it without re-creating the third angle?

18. **Technology** Is it possible for the midpoints of the three altitudes of a triangle to be collinear? Investigate by using geometry software. Write a paragraph describing your findings.

19. Sketch the section formed when the plane slices the cube as shown.

20. Use your geometry tools to draw rhombus $RHOM$ so that $HO = 6.0\ cm$ and $m\angle R = 120^\circ$.

21. Use your geometry tools to draw kite $KYTE$ so that $KY = YT = 4.8\ cm$, diagonal $YE = 6.4\ cm$, and $m\angle Y = 80^\circ$. 

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LEsson 3.7 Constructing Points of Concurrency  
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For Exercises 22–26, complete each geometric construction and name it.

22. 
23. 
24. 
25. 
26. 

**IMPROVING YOUR VISUAL THINKING SKILLS**

**The Puzzle Lock**

This mysterious pattern is a lock that must be solved like a puzzle. Here are the rules:

- You must make eight moves in the proper sequence.
- To make each move (except the last), you place a gold coin onto an empty circle, then slide it along a diagonal to another empty circle.
- You must place the first coin onto circle 1, then slide it to either circle 4 or circle 6.
- You must place the last coin onto circle 5.
- You do not slide the last coin.

Solve the puzzle. Copy and complete the table to show your solution.

<table>
<thead>
<tr>
<th>Coin</th>
<th>Placed on</th>
<th>Slid to</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Second</td>
<td></td>
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<td>Third</td>
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<td>Sixth</td>
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<tr>
<td>Seventh</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eighth</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
The universe may be as great as they say, but it wouldn't be missed if it didn't exist.
PIET HEIN

In the previous lesson you discovered that the three angle bisectors are concurrent, the three perpendicular bisectors of the sides are concurrent, and the three altitudes in a triangle are concurrent. You also discovered the properties of the incenter and the circumcenter. In this lesson you will investigate the medians of a triangle.

You may choose to do the first investigation using the Dynamic Geometry Exploration The Centroid at www.keymath.com/DG.

LESSON 3.8 The Centroid

Investigation 1
Are Medians Concurrent?

Each person in your group should draw a different triangle for this investigation. Make sure you have at least one acute triangle, one obtuse triangle, and one right triangle in your group.

On a sheet of patty paper, draw as large a scalene triangle as possible and label it CNR, as shown at right. Locate the midpoints of the three sides. Construct the medians and complete the conjecture.

Median Concurrency Conjecture
The three medians of a triangle are concurrent.

The point of concurrency of the three medians is the centroid.

Step 1
You will need
- construction tools
- geometry software (optional)

Step 2
Label the three medians CT, NO, and RE. Label the centroid D.

Step 3
Use your compass or another sheet of patty paper to investigate whether there is anything special about the centroid. Is the centroid equidistant from the three vertices? From the three sides? Is the centroid the midpoint of each median?
In earlier lessons you discovered that the midpoint of a segment is the balance point or center of gravity. You also saw that when a set of segments is arranged into a triangle, the line through each midpoint of a side and the opposite vertex can act as a line of balance for the triangle. Can you then balance a triangle on a median? Let’s take a look.

Step 1
Place your patty paper or printout from the previous investigation on a piece of mat board or cardboard. With a sharp pencil tip or compass tip, mark the three vertices, the three midpoints, and the centroid on the board.

Step 2
Draw the triangle and medians on the cardboard. Cut out the cardboard triangle.

Step 3
Try balancing the triangle on one of the three medians by placing the median on the edge of a ruler. If you are successful, what does that imply about the areas of the two triangles formed by one median? Try balancing the triangle on another median. Will it balance on each of the three medians?

Step 4
Is there a single point where you can balance the triangle?

Centroid Conjecture

The centroid of a triangle divides each median into two parts so that the distance from the centroid to the vertex is $\frac{2}{3}$ the distance from the centroid to the midpoint of the opposite side.

Investigation 2
Balancing Act

You will need
- cardboard
- a straightedge

Step 1
Place your patty paper or printout from the previous investigation on a piece of mat board or cardboard. With a sharp pencil tip or compass tip, mark the three vertices, the three midpoints, and the centroid on the board.

Step 2
Draw the triangle and medians on the cardboard. Cut out the cardboard triangle.

Step 3
Try balancing the triangle on one of the three medians by placing the median on the edge of a ruler. If you are successful, what does that imply about the areas of the two triangles formed by one median? Try balancing the triangle on another median. Will it balance on each of the three medians?

Step 4
Is there a single point where you can balance the triangle?
If you have found the balancing point for the triangle, you have found its **center of gravity**. State your discovery as a conjecture, and add it to your conjecture list.

**Center of Gravity Conjecture**

The **center** of a triangle is the center of gravity of the triangular region.

The triangle balances on each median and the centroid is on each median, so the triangle balances on the centroid. As long as the weight of the cardboard is distributed evenly throughout the triangle, you can balance any triangle at its centroid. For this reason, the centroid is a very useful point of concurrency, especially in physics.

You have discovered special properties of three of the four points of concurrency—the incenter, the circumcenter, and the centroid. The incenter is the center of an inscribed circle, the circumcenter is the center of a circumscribed circle, and the centroid is the center of gravity.

You can learn more about the orthocenter in the project *Is There More to the Orthocenter?*

In physics, the center of gravity of an object is an imaginary point where the total weight is concentrated. The center of gravity of a tennis ball, for example, would be in the hollow part, not in the actual material of the ball. The idea is useful in designing structures as complicated as bridges or as simple as furniture. Where is the center of gravity of the human body?
1. Birdy McFly is designing a large triangular hang glider. She needs to locate the center of gravity for her glider. Which point does she need to locate? Birdy wishes to decorate her glider with the largest possible circle within her large triangular hang glider. Which point of concurrency does she need to locate?

In Exercises 2–4, use your new conjectures to find each length.

2. Point $M$ is the centroid.

3. Point $G$ is the centroid.

4. Point $Z$ is the centroid.

5. **Construction** Construct an equilateral triangle, then construct angle bisectors from two vertices, medians from two vertices, and altitudes from two vertices. What can you conclude?

6. **Construction** On patty paper, draw a large isosceles triangle with an acute vertex angle that measures less than 40°. Copy it onto three other pieces of patty paper. Construct the centroid on one patty paper, the incenter on a second, the circumcenter on a third, and the orthocenter on a fourth. Record the results of all four pieces of patty paper on one piece of patty paper. What do you notice about the four points of concurrency? What is the order of the four points of concurrency from the vertex to the opposite side in an acute isosceles triangle?

7. **Technology** Use geometry software to construct a large isosceles acute triangle. Construct the four points of concurrency. Hide all constructions except for the points of concurrency. Label them. Drag until it has an obtuse vertex angle. Now what is the order of the four points of concurrency from the vertex angle to the opposite side? When did the order change? Do the four points ever become one?
8. **Mini-Investigation** Where do you think the center of gravity is located on a square? A rectangle? A rhombus? In each case the center of gravity is not that difficult to find, but what about an ordinary quadrilateral? Experiment to discover a method for finding the center of gravity for a quadrilateral by geometric construction. Test your method on a large cardboard quadrilateral.

**Review**

9. Sally Solar is the director of Lunar Planning for Galileo Station on the moon. She has been asked to locate the new food production facility so that it is equidistant from the three main lunar housing developments. Which point of concurrency does she need to locate?

10. Construct circle $O$. Place an arbitrary point $P$ within the circle. Construct the longest chord passing through $P$. Construct the shortest chord passing through $P$. How are they related?

11. A billiard ball is hit so that it travels a distance equal to $AB$ but bounces off the cushion at point $C$. Copy the figure, and sketch where the ball will rest.

12. **Application** In alkyne molecules all the bonds are single bonds except one triple bond between two carbon atoms. The first three alkynes are modeled below. The dash (--) between letters represents single bonds. The triple dash (≡) between letters represents a triple bond.

$\begin{align*}
&\text{Ethyne} \\
&C_2H_2 & \text{Propyne} \\
&C_3H_4 & \text{Butyne} \\
&C_4H_6
\end{align*}$

Sketch the alkyne with eight carbons in the chain. What is the general rule for alkynes ($C_nH_{2n}$)? In other words, if there are $n$ carbon atoms (C), how many hydrogen atoms (H) are in the alkyne?
13. When plane figure A is rotated about the line, it produces the solid figure B. What is the plane figure that produces the solid figure D?

14. Copy the diagram below. Use your Vertical Angles Conjecture and Parallel Lines Conjecture to calculate each lettered angle measure.

15. A brother and a sister have inherited a large triangular plot of land. The will states that the property is to be divided along the altitude from the northernmost point of the property. However, the property is covered with quicksand at the northern vertex. The will states that the heir who figures out how to draw the altitude without using the northern vertex point gets to choose his or her parcel first. How can the heirs construct the altitude? Is this a fair way to divide the land? Why or why not?

16. At the college dorm open house, each of the 20 dorm members invites two guests. How many greetings are possible if you do not count dorm members greeting each other?
In the previous lessons you discovered the four points of concurrency: circumcenter, incenter, orthocenter, and centroid. In this activity you will discover how these points relate to a special line, the Euler line.

The Euler line is named after the Swiss mathematician Leonhard Euler (1707–1783), who proved that three points of concurrency are collinear.

You may choose to do this activity using the Dynamic Geometry Exploration The Euler Line at www.keymath.com/DG.

Activity
Three Out of Four

You are going to look for a relationship among the points of concurrency.

Step 1
Draw a scalene triangle and have each person in your group trace the same triangle on a separate piece of patty paper.

Step 2
Have each group member construct with patty paper a different point of the four points of concurrency for the triangle.

Step 3
Record the group’s results by tracing and labeling all four points of concurrency on one of the four pieces of patty paper. What do you notice? Compare your group results with the results of other groups near you. State your discovery as a conjecture.

Euler Line Conjecture

The , , and are the three points of concurrency that always lie on a line.
The three special points that lie on the Euler line determine a segment called the **Euler segment**. The point of concurrency between the two endpoints of the Euler segment divides the segment into two smaller segments whose lengths have an exact ratio.

**Step 4**
With a compass or patty paper, compare the lengths of the two parts of the Euler segment. What is the ratio? Compare your group’s results with the results of other groups and state your conjecture.

**Euler Segment Conjecture**

The \( ? \) divides the Euler segment into two parts so that the smaller part is \( ? \) the larger part.

**Step 5**
Use your conjectures to solve this problem.

\[ AC \] is an Euler segment containing three points of concurrency, \( A, B, C \), so that \( AB > BC \).

\[ AC = 24 \text{ m}, \ AB = ?, \ BC = ? \].

---

**IS THERE MORE TO THE ORTHOCENTER?**

At this point you may still wonder what’s special about the orthocenter. It does lie on the Euler line. Is there anything else surprising or special about it?

Use geometry software to investigate the orthocenter. Construct a triangle \( ABC \) and its orthocenter \( O \). Drag a vertex of the triangle around. Where does the orthocenter lie in an acute triangle? An obtuse triangle? A right triangle?

Hide the altitudes. Draw segments from each vertex to the orthocenter shown. Now find the orthocenter of each of the three new triangles formed. What happens?

Experiment dragging the different points, and observe the relationships among the four orthocenters.

Write a paragraph about your findings, concluding with a conjecture about the orthocenter. Your project should include

- Your observations about the orthocenter, with drawings.
- Answers to all the questions above.
- A conjecture stating what’s special about the orthocenter.

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The Geometer’s Sketchpad was used to create this diagram and to hide the unnecessary lines. Using Sketchpad, you can quickly construct triangles and their points of concurrency. Once you make a conjecture, you can drag to change the shape of the triangle to see whether your conjecture is true.
In Chapter 1, you defined many terms that help establish the building blocks of geometry. In Chapter 2, you learned and practiced inductive reasoning skills. With the construction skills you learned in this chapter, you performed investigations that lay the foundation for geometry.

The investigation section of your notebook should be a detailed report of the mathematics you’ve already done. Beginning in Chapter 1 and continuing in this chapter, you summarized your work in the definition list and the conjecture list. Before you begin the review exercises, make sure your conjecture list is complete. Do you understand each conjecture? Can you draw a clear diagram that demonstrates your understanding of each definition and conjecture? Can you explain them to others? Can you use them to solve geometry problems?

**EXERCISES**

For Exercises 1–10, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.

1. In a geometric construction, you use a protractor and a ruler.
2. A diagonal is a line segment in a polygon that connects any two vertices.
3. A trapezoid is a quadrilateral with exactly one pair of parallel sides.
4. A square is a rhombus with all angles congruent.
5. If a point is equidistant from the endpoints of a segment, then it must be the midpoint of the segment.
6. The set of all the points in the plane that are a given distance from a line segment is a pair of lines parallel to the given segment.
7. It is not possible for a trapezoid to have three congruent sides.
8. The incenter of a triangle is the point of intersection of the three angle bisectors.
9. The orthocenter of a triangle is the point of intersection of the three altitudes.
10. The incenter, the centroid, and the orthocenter are always inside the triangle.

For Exercises 11–18, match each geometric construction with one of the figures below.

11. Construction of a midsegment
12. Construction of an altitude
13. Construction of a centroid in a triangle
14. Construction of an incenter
15. Construction of an orthocenter in a triangle
16. Construction of a circumcenter
17. Construction of an equilateral triangle
18. Construction of an angle bisector

Construction For Exercises 19–24, perform a construction with compass and straightedge or with patty paper. Choose the method for each problem, but do not mix the tools in any one problem. In other words, play each construction game fairly.

19. Draw an angle and construct a duplicate of it.
20. Draw a line segment and construct its perpendicular bisector.
21. Draw a line and a point not on the line. Construct a perpendicular to the line through the point.

22. Draw an angle and bisect it.

23. Construct an angle that measures 22.5°.

24. Draw a line and a point not on the line. Construct a second line so that it passes through the point and is parallel to the first line.

25. Brad and Janet are building a home for their pet hamsters, Riff and Raff, in the shape of a triangular prism. Which point of concurrency in the triangular base do they need to locate in order to construct the largest possible circular entrance?

26. Adventurer Dakota Davis has a map that once showed the location of a large bag of gold. Unfortunately, the part of the map that showed the precise location of the gold has burned away. Dakota visits the area shown on the map anyway, hoping to find clues. To his surprise, he finds three headstones with geometric symbols on them. The clues lead him to think that the treasure is buried at a point equidistant from the three stones. If Dakota’s theory is correct, how should he go about locating the point where the bag of gold might be buried?

Construction For Exercises 27–32, use the given segments and angles to construct each figure. The lowercase letter above each segment represents the length of the segment.

27. △ABC given ∠A, ∠C, and AC = z

28. A segment with length 2y + x − ½z

29. △PQR with PQ = 3x, QR = 4x, and PR = 5x

30. Isosceles triangle ABD given ∠A, and AB = BD = 2y

31. Quadrilateral ABFD with m∠A = m∠B, AD = BF = y, and AB = 4x

32. Right triangle TRI with hypotenuse TI, TR = x, and RI = y, and a square on TI, with TI as one side.
Mixed Review

Tell whether each symbol in Exercises 33–36 has reflectional symmetry, rotational symmetry, neither, or both. (The symbols are used in meteorology to show weather conditions.)

33.
34.
35.
36.

For Exercises 37–40, match the term with its construction.

37. Centroid
38. Circumcenter
39. Incenter
40. Orthocenter

A.
B.
C.
D.

For Exercises 41–54, identify the statement as true or false. For each false statement, explain why it is false or sketch a counterexample.

41. An isosceles right triangle is a triangle with an angle measuring 90° and no two sides congruent.

42. If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

43. An altitude of a triangle must be inside the triangle.

44. The orthocenter of a triangle is the point of intersection of the three perpendicular bisectors of the sides.

45. If two lines are parallel to the same line, then they are parallel to each other.

46. If the sum of the measure of two angles is 180°, then the two angles are vertical angles.

47. Any two consecutive sides of a kite are congruent.

48. If a polygon has two pairs of parallel sides, then it is a parallelogram.

49. The measure of an arc is equal to one half the measure of its central angle.

50. If TR is a median of △TIE and point D is the centroid, then TD = 3DR.

51. The shortest chord of a circle is the radius of a circle.

52. An obtuse triangle is a triangle that has one angle with measure greater than 90°.
53. Inductive reasoning is the process of showing that certain statements follow logically from accepted truths.

54. There are exactly three true statements in Exercises 41–54.

55. In the diagram, $p \parallel q$.
   a. Name a pair of corresponding angles.
   b. Name a pair of alternate exterior angles.
   c. If $m \angle 3 = 42^\circ$, what is $m \angle 6$?

In Exercises 56 and 57, use inductive reasoning to find the next number or shape in the pattern.

56. 100, 97, 91, 82, 70

57. [diagram with numbers and shapes]

58. Consider the statement “If the month is October, then the month has 31 days.”
   a. Is the statement true?
   b. Write the converse of this statement.
   c. Is the converse true?

59. Find the point on the cushion at which a pool player should aim so that the white ball will hit the cushion and pass over point $Q$.

For Exercises 60 and 61, find the function rule for the sequence. Then find the 20th term.

60. | $n$ | 1 | 2 | 3 | 4 | 5 | 6 | ... | $n$ | ... | 20 |
   | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
   | $f(n)$ | -1 | 2 | 5 | 8 | 11 | 14 | ... | ... | ... | ...

61. | $n$ | 1 | 2 | 3 | 4 | 5 | 6 | ... | $n$ | ... | 20 |
   | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
   | $f(n)$ | 0 | 3 | 8 | 15 | 24 | 35 | ... | ... | ... | ...

62. Developing Proof Calculate each lettered angle measure. Explain how you determined the measures $c$ and $f$. [diagram with angles labeled]
63. Draw a scalene triangle $ABC$. Use a straightedge and compass to construct the incenter of $\triangle ABC$.

64. **Developing Proof** What’s wrong with this picture?

65. What is the minimum number of regions that are formed by 100 distinct lines in a plane? What is the maximum number of regions formed by 100 lines in the plane?

---

**Assessing What You’ve Learned**

The subject of this chapter was the tools of geometry, so assessing what you’ve learned really means assessing what you can do with those tools. Can you do all the constructions you learned in this chapter? Can you show how you arrived at each conjecture? Demonstrating that you can do tasks like these is sometimes called performance assessment.

Look over the constructions in the Chapter Review. Practice doing any of the constructions that you’re not absolutely sure of. Can you do each construction using either compass and straightedge or patty paper? Look over your conjecture list. Can you perform all the investigations that led to these conjectures?

Let a friend, classmate, teacher, or family member choose one construction and one investigation for you to demonstrate. Do every step from start to finish, and explain what you’re doing.

**ORGANIZE YOUR NOTEBOOK**

- Your notebook should have an investigation section, a definition list, and a conjecture list. Review the contents of these sections. Make sure they are complete, correct, and well organized.
- Write a one-page chapter summary from your notes.

**WRITE IN YOUR JOURNAL**

How does the way you are learning geometry—doing constructions, looking for patterns, and making conjectures—compare to the way you’ve learned math in the past?

**UPDATE YOUR PORTFOLIO**

Choose a construction problem from this chapter that you found particularly interesting and/or challenging. Describe each step, including how you figured out how to move on to the next step. Add this to your portfolio.